

THE SIZES OF THE INTERSECTIONS OF TWO UNITALS IN $\text{PG}(2, q^2)$

DAVID B. CHANDLER*

ABSTRACT. We show that the size of the intersection of a Hermitian variety in $\text{PG}(n, q^2)$, and any set satisfying an r -dimensional-subspace intersection property, is congruent to 1 modulo a power of p . In particular, in the case where $n = 2$, if the two sets are a Hermitian unital and any other unital, the size of the intersection is congruent to 1 modulo \sqrt{q} or modulo \sqrt{pq} . If the second unital is a Buekenhout-Metz unital, we show that the size is congruent to 1 modulo q .

1. INTRODUCTION

A *unital* is any 2-design with parameters of the form $(n^3 + 1, n + 1, 1)$. That is, we have a set \mathcal{P} of $v = n^3 + 1$ points and a collection \mathcal{B} , the blocks, of subsets of \mathcal{P} , having the following two properties: Each block has size $k = n + 1$; and any two points are jointly contained in exactly one block. A unital U is *embedded* in a finite projective plane of order q^2 if it is a set of $q^3 + 1$ points of the plane with the property that every line of the plane intersects exactly 1 or $q + 1$ points of U . In this paper we are interested in unitals which are embedded in $\text{PG}(2, q^2)$, $q = p^t$, for some prime p .

The classical or *Hermitian* unital is a Hermitian variety, the set of zeroes of a unitary form. Such forms on $\mathbb{F}_{q^2}^3$ are projectively equivalent to

$$x^{q+1} + y^{q+1} + z^{q+1}$$

over \mathbb{F}_{q^2} , the field with q^2 elements. More generally, we are also interested in Hermitian varieties in $\text{PG}(n, q^2)$, which are projectively equivalent to $\sum_{i=0}^n x_i^{q+1} = 0$.

Buekenhout and Metz proved the existence of nonclassical unitals in the 1970s [10]. Around 1990 Baker and Ebert generalized Buekenhout's and Metz's construction to describe a two-parameter family of unitals $U_{a,b} = \{(0, 1, 0)\} \cup \{(x, ax^2 + bx^{q+1} + r, 1) : x \in \mathbb{F}_{q^2} \text{ and } r \in \mathbb{F}_q\}$, in

Key words and phrases. unital, Hermitian variety, incidence matrix, p -adic ring.

*This work was partially supported by NSF grants DMS 0701049 and DMS 1001557.

any desarguesian plane of square order, where $a, b \in \mathbb{F}_{q^2}$ meet some condition and $q > 2$ [2, 1, 7]. We call these unitals *Buekenhout-Metz unitals*, or *B-M unitals*. A B-M unital is Hermitian if and only if $a = 0$. The construction is summarized in Section 3.

When q is an odd power of 2, there is one nonclassical ovoid known in $\text{PG}(3, q)$, the *Tits ovoid*. By replacing the elliptic quadric in the B-M construction by a Tits ovoid, we get one more projective equivalence class of unitals in $\text{PG}(2, q^2)$. At present there are no other unitals known in desarguesian planes. For more information about unitals, we refer the reader to [4].

Kestenband [8] showed that if H_1 and H_2 are Hermitian unitals in $\text{PG}(2, q^2)$, then $|H_1 \cap H_2| \in \{1, q+1, q^2-q+1, q^2+1, q^2+q+1, q^2+2q+1\}$, and also determined the possible intersection configurations. Note that the sizes are all congruent to 1 modulo q . Baker and Ebert [2] then proved that the size of the intersection of a Hermitian unital with the special type of B-M unital having $b = 0$ is congruent to 1 modulo p . They also conjectured that the size of the intersection of any unital with a Hermitian unital would turn out to be congruent to 1 modulo q . Blokhuis, Brouwer, and Wilbrink [5] soon proved that the size of this intersection is congruent to 1 modulo p , by showing that the Hermitian unital is in the code of lines of $\text{PG}(2, q^2)$. In this article we prove the following theorem.

Theorem 1. *Let H be a Hermitian unital embedded in $\text{PG}(2, q^2)$ and let U be any other unital embedded in $\text{PG}(2, q^2)$, where $q = p^t$, and p is a prime. Then*

$$|H \cap U| \equiv 1 \pmod{p^{\lceil t/2 \rceil}}.$$

Moreover, if U is a Buekenhout-Metz unital, then

$$|H \cap U| \equiv 1 \pmod{q}.$$

We should note that if neither U_1 nor U_2 is Hermitian, then nothing particular can be said about $|U_1 \cap U_2|$. A computer check of the intersection sizes of pairs randomly chosen from the known non-Hermitian unitals reveals no particular pattern.

2. HERMITIAN vs. ARBITRARY UNITAL

In this section we obtain the first part of Theorem 1 as a corollary of a more general result.

Let p be a prime, let $q = p^t$, and let V be an $(n + 1)$ -dimensional vector space over \mathbb{F}_{q^2} with coordinate functions x_0, \dots, x_n . We denote the set of projective points of $\text{PG}(n, q^2)$, i.e., one-dimensional subspaces

of V , by \mathcal{L}_1 , and the set of projective $(r - 1)$ -dimensional subspaces $\text{PG}(n, q^2)$, *i.e.*, r -dimensional subspaces of V , by \mathcal{L}_r .

Let H be a Hermitian variety of $\text{PG}(n, q^2)$. Note that every Hermitian variety is projectively equivalent to the zeroes of

$$\sum_{i=0}^n x_i^{q+1}.$$

Suppose we have r , a vector-space dimension, $1 < r \leq n$, another positive integer β , and a set of points $S \subset \mathcal{L}_1$ with the following intersection property:

Property I. Every element of \mathcal{L}_r meets S in a number of points which is divisible by p^β .

Then we will prove that $|S \cap H|$ is divisible by a certain power of p . Note that from the size of a projective $(r - 1)$ -space,

$$\beta \leq 2t(r - 1) \leq 2t(n - 1).$$

Definition 2. Let $A_{r,1}$ be the $(0, 1)$ -matrix, columns indexed by \mathcal{L}_1 , the points, and rows by \mathcal{L}_r , the projective $(r - 1)$ -spaces, whose entries are

$$a_{Y,Z} = \begin{cases} 1, & \text{if } Z \subset Y; \\ 0, & \text{otherwise;} \end{cases} \quad Y \in \mathcal{L}_r, \quad Z \in \mathcal{L}_1.$$

We call $A_{r,1}$ the incidence matrix between \mathcal{L}_r and \mathcal{L}_1 .

Consider $\mathbb{F}_{q^2}^{\mathcal{L}_1}$, the space of \mathbb{F}_{q^2} -valued functions on \mathcal{L}_1 . A useful basis for this space is given by the set of monomials [3]:

$$\overline{\mathcal{B}} = \left\{ \prod_{i=0}^n x_i^{b_i}, \quad \begin{array}{l} 0 \leq b_i \leq q^2 - 1, \quad 0 \leq i \leq n, \quad (q^2 - 1) \mid \sum_{i=0}^n b_i, \\ (b_0, \dots, b_n) \neq (q^2 - 1, \dots, q^2 - 1) \end{array} \right\}.$$

It will be helpful to view the entries of $A_{r,1}$ as coming from some p -adic local ring. Let $q = p^t$ and let $K = \mathbb{Q}_p(\xi_{q-1})$ be the unique unramified extension of degree t over \mathbb{Q}_p , the field of p -adic numbers, where ξ_{q-1} is a primitive $(q - 1)^{\text{th}}$ root of unity in K . Let $R = \mathbb{Z}_p[\xi_{q-1}]$ be the ring of integers in K and let \mathfrak{p} be the unique maximal ideal in R . Then the reduction of $R \pmod{\mathfrak{p}}$ will be \mathbb{F}_q . Define \bar{x} to be $x \pmod{\mathfrak{p}}$ for $x \in R$. Let T_q be the set of roots of $x^q = x$ in R (a Teichmüller set) and let T be the Teichmüller character of \mathbb{F}_q , so that $T(\bar{x}) = x$ for $x \in T_q$. We adopt the convention that $T^0(\bar{x}) = 1$, $\bar{x} \in \mathbb{F}_q^*$, while $T^{q-1}(0) = 0$, and $T^{q-1}(\bar{x}) = 1$, $\bar{x} \in \mathbb{F}_q^*$. We will use T to lift a basis of $\mathbb{F}_q^{\mathcal{L}_1}$ to a basis of $R^{\mathcal{L}_1}$.

For any $u \in R$, we define $\nu_p(u)$ to be the p -adic valuation of u . That is, $\nu_p(u) = \alpha$ if $p^\alpha \mid u$ but $p^{\alpha+1} \nmid u$.

We obtain a *lifted* basis \mathcal{B} for the free module $R^{\mathcal{L}_1}$ (see [6] for proof):

$$\mathcal{B} = \{T(\prod x_i^{b_i}) \mid \prod x_i^{b_i} \in \overline{\mathcal{B}}\}.$$

The matrix $A_{r,1}$ can be viewed as a map from $\mathbb{F}_{q^2}^{\mathcal{L}_1}$ to $\mathbb{F}_{q^2}^{\mathcal{L}_r}$ (or *vice versa*). For instance, let \mathbf{u} be the column $(0,1)$ -characteristic vector of a point set. Then $A_{r,1}\mathbf{u}$ records the number of points in each $(r-1)$ -dimensional subspace of $\text{PG}(n, q^2)$.

In [6] it was shown that \mathcal{B} forms a “Smith normal form” basis for the map from \mathcal{L}_1 to \mathcal{L}_r : Let \mathbf{v} be the column vector representing an element of \mathcal{B} . Then $A_{r,1}\mathbf{v}$ is the corresponding invariant (a power of p) multiplied by an integral vector indexed by \mathcal{L}_r .

We recall the formula for the invariants (stated for \mathbb{F}_{q^2}). To each nonconstant basis monomial $f = x_0^{b_0} \cdots x_n^{b_n} \in \overline{\mathcal{B}}$, we associate a pair of $2t$ -tuples, $(\lambda_0, \dots, \lambda_{2t-1})$ (called the *type*), and (s_0, \dots, s_{2t-1}) (called the \mathcal{H} -*type*). The type of $T(f)$ is that of f . We expand each exponent as

$$\begin{aligned} b_i &= a_{i,0} + a_{i,1}p + \cdots + a_{i,2t-1}p^{2t-1}; & 0 \leq i \leq n, \\ && 0 \leq j \leq 2t-1. \end{aligned}$$

Then we define

$$\lambda_j = a_{0,j} + a_{1,j} + \cdots + a_{n,j} \quad (1)$$

$$\begin{aligned} s_j &= \frac{1}{q^2 - 1} \sum_{i=0}^n \left(\sum_{\ell=0}^{j-1} p^{\ell+2t-j} a_{i,\ell} + \sum_{\ell=j}^{2t-1} p^{\ell-j} a_{i,\ell} \right) \\ &= \frac{1}{q^2 - 1} \sum_{i=0}^n (p^{2t-j} b_i \pmod{q^2 - 1}) \end{aligned} \quad (2)$$

and we have the relation $\lambda_j = ps_{j+1} - s_j$ (subscripts modulo $2t$). The numbers $(q^2 - 1)s_j$ are called the *twisted degrees* of f . The formula we want is given as follows.

Proposition 3 ([6]). *Let $f \in \mathcal{B}$ be a basis monomial. If $f = 1$, the corresponding p -adic invariant is 1. Otherwise let the \mathcal{H} -type of f be (s_0, \dots, s_{2t-1}) . Then the corresponding p -adic invariant for the map $A_{r,1}$ from $\mathbb{F}_{q^2}^{\mathcal{L}_1}$ to $\mathbb{F}_{q^2}^{\mathcal{L}_r}$ is given by p^α , where*

$$\alpha = \sum_{j=0}^{2t-1} \max\{0, r - s_j\}.$$

For any integer u , we also define $\sigma_p(u)$ to be the p -adic digit sum. That is, if

$$u = \sum_{j=0}^m u_j p^j, \quad 0 \leq u_j \leq p-1, \quad 0 \leq j \leq m, \quad \text{then} \quad \sigma_p(u) = \sum_{j=0}^m u_j.$$

Note that $\sigma_p(u_1 u_2) \leq \sigma_p(u_1) \sigma_p(u_2)$, $\sigma_p(u_1 + u_2) \leq \sigma_p(u_1) + \sigma_p(u_2)$, and

$$\sum_{i=0}^n \sigma_p(b_i) = \sum_{j=0}^{2t-1} \lambda_j = (p-1) \sum_{j=1}^{2t-1} s_j \quad (3)$$

for $f = x_0^{b_0} \cdots x_n^{b_n}$ of type $(\lambda_0, \dots, \lambda_{2t-1})$. Here, b_i is taken as the least positive residue modulo $(q^2 - 1)$, unless it is already zero. Note that reduction to the least positive residue may reduce $\sigma_p(b_i)$, but never increase it.

We use the fact that

$$T(a+b) \equiv (T(a) + T(b))^{q^\ell} \pmod{q^\ell}, \quad \text{for } a, b \in \mathbb{F}_q,$$

for any positive integer ℓ , which enables us to bring the Teichmüller character inside the parentheses [6].

Let H be the Hermitian variety given by

$$\sum_{i=0}^n x_i^{q+1} = 0.$$

Then we take \overline{H} as the complement of H in $\mathrm{PG}(n, q^2)$. Consider the characteristic function:

$$\begin{aligned} \chi_{\overline{H}} : \mathcal{L}_1 &\rightarrow \{0, 1\} \subset R \\ \chi_{\overline{H}}(\langle(x_0, x_1, \dots, x_n)\rangle) &= T \left(\sum_{i=0}^n x_i^{q+1} \right)^{q-1} \\ &\equiv \left(\sum_{i=0}^n T(x_i)^{q+1} \right)^{q^{2\ell+1}-q^{2\ell}} \pmod{q^{2\ell}}. \end{aligned} \quad (4)$$

We see that each term in the expansion of the RHS of (4) has the form

$$f = \binom{q^{2\ell+1} - q^{2\ell}}{k_0, \dots, k_n} \prod_{i=0}^n T(x_i)^{(q+1)k_i}$$

where $\sum_{i=0}^n k_i = q^{2\ell+1} - q^{2\ell}$. From Legendre's formula, $\nu_p(n!) = \frac{n-\sigma_p(n)}{p-1}$, we have

$$(p-1)\nu_p \binom{q^{2\ell+1} - q^{2\ell}}{k_0, \dots, k_n} = \sum_{i=0}^n \sigma_p(k_i) - (p-1)t.$$

Since

$$2\sigma_p(k_i) \geq \sigma_p((q+1)k_i) \geq \sigma_p(b_i),$$

where b_i is the least positive residue of $(q+1)k_i$ modulo (q^2-1) , if $k_i > 0$, we substitute from (3) to obtain

$$2\nu_p\binom{q^{2\ell+1}-q^{2\ell}}{k_0, \dots, k_n} \geq \frac{1}{p-1} \sum_{j=0}^{2t-1} \lambda_j - 2t = \sum_{j=0}^{2t-1} s_j - 2t \quad (5)$$

where $(\lambda_0, \dots, \lambda_{2t-1})$ and (s_0, \dots, s_{2t-1}) are the type and \mathcal{H} -type of f . In the special case where $b_0 = b_1 = \dots = b_n = (q^2-1)$, the \mathcal{H} -type is actually not defined in [3], but the same calculation shows that q^n divides the multinomial coefficient in this case (as if the \mathcal{H} -type were $(n+1, \dots, n+1)$).

We now consider the set S with Property I. Let

$$\begin{aligned} \chi_S : \mathcal{L}_1 &\rightarrow \{0, 1\} \subset R \\ \chi_S(\langle(x_0, x_1, \dots, x_n)\rangle) &= \chi_S(x_0, x_1, \dots, x_n) \\ \chi_S(\langle(0, \dots, 0)\rangle) &= \chi_S(0, \dots, 0) = 0 \end{aligned}$$

be the characteristic function of S expressed as a polynomial in $T(x_0), \dots, T(x_n)$. We now assume that χ_S is restricted to points of V other than the origin and use the identity,

$$\prod_{i=0}^n (1 - x_i^{q^2-1}) = 0, \quad (6)$$

to eliminate the monomial $\prod_{i=0}^n x_i^{q^2-1}$. Let $g = c_S \prod_{i=0}^n T(x_i)^{b'_i}$ be a monomial term of χ_S , where $c_S \in R$, and the \mathcal{H} -type of g be (s'_0, \dots, s'_{2t-1}) . Since the monomials form a Smith-normal-form basis for the incidence matrix $A_{r,1}$, our divisibility property (*i.e.*, Property I) implies that p^β must divide the product of c_S and the invariant corresponding to g (for the matrix $A_{r,1}$). Thus p^β divides the constant term, and for each nonconstant term g , we get

$$\nu_p(c_S) + \sum_{j=0}^{2t-1} \max\{0, r - s'_j\} \geq \beta,$$

or

$$\nu_p(c_S) \geq \max \left\{ 0, \beta - 2rt + \sum_{j=0}^{2t-1} \min\{r, s'_j\} \right\}. \quad (7)$$

Observe that

$$|S \cap \overline{H}| = \frac{1}{q^2-1} \sum_{\mathbf{x}=(x_0, x_1, \dots, x_n) \in V} \chi_S(\mathbf{x}) \chi_{\overline{H}}(\mathbf{x}),$$

since the function we have for $\chi_{\overline{H}}$ evaluates to zero at the origin. Let

$$f = c_{\overline{H}} \prod_{i=0}^n T(x_i)^{b_i}, \quad c_{\overline{H}} = \binom{q^{\ell+1} - q^\ell}{k_0, \dots, k_n},$$

be some term in the expansion of the RHS of (4) with \mathcal{H} -type (s_0, \dots, s_{2t-1}) , and let

$$g = c_S \prod_{i=0}^n T(x_i)^{b'_i}$$

be some monomial term of χ_S with \mathcal{H} -type (s'_0, \dots, s'_{2t-1}) . We wish to show that a certain power of p always divides

$$\begin{aligned} \sum_{\mathbf{x} \in V} fg &= c_{\overline{H}} c_S \sum_{\mathbf{x} \in V} \prod_{i=0}^n T(x_i)^{b_i + b'_i} \\ &= c_{\overline{H}} c_S \prod_{i=0}^n \sum_{x_i \in \mathbb{F}_{q^2}} T(x_i)^{b_i + b'_i}. \end{aligned}$$

Since

$$\sum_{x \in \mathbb{F}_{q^2}} T(x)^j = \begin{cases} q^2, & \text{if } j = 0, \\ 0, & \text{if } j \not\equiv 0 \pmod{q^2 - 1}, \\ q^2 - 1, & \text{if } j > 0, j \equiv 0 \pmod{q^2 - 1}, \end{cases} \quad (8)$$

we only need to consider terms with $b_i + b'_i \in \{0, q^2 - 1, 2(q^2 - 1)\}$, $0 \leq i \leq n$. We first suppose that $b_i + b'_i = q^2 - 1$ for $0 \leq i \leq n$. In this case, fg has total degree $(q^2 - 1)(n + 1)$, and all the twisted degrees also are $(q^2 - 1)(n + 1)$. Therefore $s_j + s'_j = n + 1$, $0 \leq j \leq 2t - 1$. Now,

$$\begin{aligned} 2\nu_p(c_{\overline{H}} c_S) &\geq \sum_{j=0}^{2t-1} (s_j - 1) + 2 \max \left\{ 0, \beta - 2rt + \sum_{j=0}^{2t-1} \min\{r, s'_j\} \right\} \quad (9) \\ &= \sum_{j=0}^{2t-1} (n - s'_j) + \max \left\{ 0, 2\beta - 4rt + 2 \sum_{j=0}^{2t-1} \min\{r, s'_j\} \right\}. \end{aligned}$$

Notice that increasing one s'_j by one decreases the sum by one if $s'_j \geq r$ or if the second term inside the “max” function of (9) is negative, and it increases the sum by one otherwise. Therefore the minimum value of the sum can be achieved by choosing all the s'_j to be either 1 or n , except for one value, say s'_0 , to be in the range $1 < s'_0 < r$. If $r - s'_j \leq n - r$, the sum does not increase if we change the value to $s'_j = n$. Therefore, if $r \leq (n + 1)/2$, the right-hand side is minimized if we always pick $s'_j = n$. If $r > (n + 1)/2$, the right-hand side is

minimized in one of two ways: either we choose $s'_j \in \{1, n\}$, $j > 0$, and s'_0 as necessary to make the second term on the RHS equal to zero, or if $s'_0 \geq 2r - n$ in that case, we also make $s'_0 = n$.

We also consider the cases where $b_i + b'_i \in \{0, 2(q^2 - 1)\}$ for some i . If $b_i = b'_i = 0$, then $s_j + s'_j$ is reduced by 1 for each j , $0 \leq j \leq 2t - 1$, compared to the case where $b_i = q^2 - 1$ and everything else is the same, which reduces our estimate for $2\nu_p(c_{\overline{H}}c_S)$ by $2t$, but from (8), we have an extra factor of q^2 , which increases our estimate by $4t$. If $b_i = b'_i = q^2 - 1$, then each s_j is increased by 1, compared to the case where $b_i = 0$, which only increases our estimate.

Summarizing the results for $r > (n + 1)/2$, we get the smallest estimate by choosing the number of j such that $s'_j = 1$ to be $\alpha = \lfloor \frac{\beta}{r-1} \rfloor$ and let $\gamma = \beta - (r - 1)\alpha$. Then $s'_0 \in \{r - \gamma, n\}$.

Since Property I also implies $\nu_p(|S|) \geq \beta$, we have proved the following:

Theorem 4. *Let $S \subset \mathcal{L}_1$ be a set of points with Property I, and let $H \subset \mathcal{L}_1$ be the point set of a nondegenerate Hermitian variety. Let α and γ be as above and let*

$$\theta = \begin{cases} \beta, & \text{if } r \leq (n + 1)/2; \\ \left\lceil \frac{(n-1)\alpha}{2} + \min \left\{ \frac{n-r+\gamma}{2}, \gamma \right\} \right\rceil, & \text{if } r > (n + 1)/2. \end{cases}$$

Then

$$p^\theta \mid |S \cap H|.$$

We note that θ is approximately $\frac{\beta(n-1)}{2(r-1)}$ if $r > (n + 1)/2$.

Taking $n = r = 2$, $\beta = t$, and $S = \overline{U}$, we have:

Corollary 5. *Let H be a Hermitian unital and let U be an arbitrary unital in $\mathrm{PG}(2, q^2)$, $q = p^t$. Then $\nu_p(|H \cap U| - 1) \geq t/2$.*

3. HERMITIAN vs. BUEKENHOUT-METZ UNITAL

In this section we show that the number of points in the intersection of a Hermitian unital and a Buekenhout-Metz unital is always congruent to 1 (modulo q). The Buekenhout-Metz construction goes as follows (see [10, 7, 1]). Start with an elliptic quadric in $A \cong \mathrm{PG}(3, q) \subset \mathrm{PG}(4, q)$ and a regular spread in $B \cong \mathrm{PG}(3, q) \subset \mathrm{PG}(4, q)$ such that the plane $A \cap B$ is a tangent plane to the quadric. Choose a point P on the same spread line as the point of tangency of $A \cap B$ to the quadric and let U^* be the cone of P and the quadric. Now construct a new plane of order q^2 , taking as points the points of $\mathrm{PG}(4, q) \setminus B$ as well as the spread lines covering B , and taking as lines the planes of

$\text{PG}(4, q)$ whose intersection with B is a spread line, as well as B itself. With inclusion as incidence, we always get a translation plane (having a regular automorphism group on the image of $\text{PG}(4, q) \setminus B$), and since we took a regular spread we get the desarguesian plane $\text{PG}(2, q^2)$. Furthermore, the cone U^* contains q^3 affine points and a spread line. The image $U \subset \text{PG}(2, q^2)$ of U^* is easily shown to be a unital.

Every unital produced by the Buekenhout-Metz construction is projectively equivalent to

$$U_{\alpha, \beta} = \{(1, y, \alpha y^2 + \beta y^{q+1} + r) \mid y \in \mathbb{F}_{q^2}, r \in \mathbb{F}_q\} \cup \{(0, 0, 1)\}.$$

If $\alpha = 0$ and $\beta \notin \mathbb{F}_q$ then $U_{0, \beta}$ is a Hermitian unital. If $\alpha \neq 0$ then $U_{\alpha, \beta}$ is a unital if and only if the following condition holds:

$$\begin{aligned} (\beta^q - \beta)^2 + 4\alpha^{q+1} &\text{ is a nonsquare of } \mathbb{F}_q, & \text{if } q \text{ is odd;} \\ \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2} \left(\frac{\beta^q + \beta}{\alpha^{q+1}} \right) &= 1, & \text{if } q \text{ is even.} \end{aligned}$$

From $z - (\alpha y^2 + \beta y^{q+1}) = r \in \mathbb{F}_q$ (if $x = 1$) we get the equation

$$\alpha^q y^{2q} - \alpha y^2 + (\beta^q - \beta) y^{q+1} - z^q + z = r - r^q = 0 \quad (10)$$

which is satisfied by the affine points of $U_{\alpha, \beta}$. The homogeneous equation for the affine points is

$$\alpha^q x^2 y^{2q} - \alpha x^{2q} y^2 + (\beta^q - \beta) x^{q+1} y^{q+1} - x^{q+2} z^q + x^{2q+1} z = 0.$$

Note that if q is even, then the left-hand side of (10) is in \mathbb{F}_q , while if q is odd, then the square of the left-hand side of (10) is in \mathbb{F}_q . Therefore in either case, raising to the $2(q-1)$ power, we get the characteristic function over \mathbb{F}_{q^2} for the affine points of the complement of $U_{\alpha, \beta}$. The complete characteristic function of the complement of $U_{\alpha, \beta}$, including all the points at infinity other than $(0, 0, 1)$, is

$$\begin{aligned} &(\alpha^q x^2 y^{2q} - \alpha x^{2q} y^2 + (\beta^q - \beta) x^{q+1} y^{q+1} - x^{q+2} z^q + x^{2q+1} z)^{2(q-1)} \\ &+ (1 - x^{q^2-1}) y^{q^2-1}. \end{aligned}$$

Taking the p -adic Teichmüller character (modulo q^2) of each term inside the parentheses and further raising the affine part of the characteristic function to the q^2 power, we finally get the characteristic function $\chi_{\overline{U}}$ in the Galois ring R/q^2R (*cf.* [11, 6]).

We also need the characteristic function of the Hermitian unital. Previously the orientation of the non-Hermitian unital was arbitrary; so we could take the simplest form of the equation for the Hermitian

unital. Now that the orientation of the B.-M. unital is fixed, we must consider a general form for the Hermitian unital:

$$\begin{aligned}
 & (c_{1,1}x + c_{1,2}y + c_{1,3}z)^{q+1} + (c_{2,1}x + c_{2,2}y + c_{2,3}z)^{q+1} \\
 & \quad + (c_{3,1}x + c_{3,2}y + c_{3,3}z)^{q+1} = \\
 & (c_{1,1}^q x^q + c_{1,2}^q y^q + c_{1,3}^q z^q) \cdot (c_{1,1}x + c_{1,2}y + c_{1,3}z) \\
 & + (c_{2,1}^q x^q + c_{2,2}^q y^q + c_{2,3}^q z^q) \cdot (c_{2,1}x + c_{2,2}y + c_{2,3}z) \\
 & + (c_{3,1}^q x^q + c_{3,2}^q y^q + c_{3,3}^q z^q) \cdot (c_{3,1}x + c_{3,2}y + c_{3,3}z) = 0,
 \end{aligned} \tag{11}$$

where the coefficients $(\mathbf{c})_{(3 \times 3)}$ form a nonsingular matrix over \mathbb{F}_{q^2} . Again we take the Teichmüller character of x, y, z , and (\mathbf{c}) and raise to the $(q^3 - q^2)$ power (since $q + 1$ powers of \mathbb{F}_{q^2} elements are already in \mathbb{F}_q) to get $\chi_{\overline{H}}$, the characteristic function of the complement of the Hermitian unital in the ring R/q^2R .

Our goal is to show that q divides

$$\sum_{(x,y,z) \in \mathbb{F}_{q^2}^3} \chi_{\overline{H}}(x, y, z) \chi_{\overline{U}}(x, y, z) = (q^2 - 1) |\overline{H} \cap \overline{U}|.$$

As before, we do so by considering the expansion of the product. In view of (8), we only need to consider terms which arise as the product of a monomial term $f = c_H T(x^{b_1} y^{b_2} z^{b_3})$ from the expansion of $\chi_{\overline{H}}$ and a term $g = c_U T(x^{b'_1} y^{b'_2} z^{b'_3})$ from the expansion of $\chi_{\overline{U}}$, satisfying $b_i + b'_i \in \{0, q^2 - 1, 2(q^2 - 1)\}$, $i = 1, 2, 3$.

If $b_i = b'_i = 0$ for some i , we already get a factor of q^2 from (8). Otherwise, if $b_i = b'_i = (q^2 - 1)$ for some i , then the twisted degrees sum to at least $4(q^2 - 1)$, and either one of f or g is $T(xyz)^{q^2-1}$, or both f and g are of \mathcal{H} -type $(2, \dots, 2)$ (if, for instance, $b_1 = b'_1 = q^2 - 1$, $b_2 = b'_2$, $b_3 = b'_3$, and $b_2 + b_3 = q^2 - 1$). In the first case, the coefficient of $T(xyz)^{q^2-1}$ is divisible by q (eliminating $T(xyz)^{q^2-1}$ using (6) makes this coefficient the constant term). In the second case, both coefficients are divisible by q (see (7) with $\beta = t$, $r = 2$, and $s_j = 2$, $0 \leq j < 2t$). Thus we assume $b_i + b'_i = q^2 - 1$.

Lemma 6. *Let (s_0, \dots, s_{2t-1}) and (s'_0, \dots, s'_{2t-1}) be the \mathcal{H} -types of the two monomials f and g described above, such that $b_i + b'_i = q^2 - 1$, $i = 1, 2, 3$. Then*

$$s_j + s'_j = 3, \quad 0 \leq j \leq 2t - 1.$$

Proof: The corresponding twisted degrees of f and g always sum to $3(q^2 - 1)$, because the degree of fg is invariant under Frobenius twisting. \square

Unlike the situation in Section 2, we consider the relation between the coefficients c_H and c_U and the shapes of the \mathcal{H} -types f and g . Note that the \mathcal{H} -type is a $2t$ -tuple consisting of 1's and 2's.

Lemma 7. *Let the \mathcal{H} -type of f_H be (s_0, \dots, s_{2t-1}) and let*

$$\xi = |\{j \mid 0 \leq j \leq t-1, s_j = 1, \text{ and } s_{j+t} = 1\}|.$$

Then c_H is divisible by $p^{t-\xi}$.

Proof: We expand (11) to get 9 terms, each of the form $v_i = c\mu\phi^q$, $i \in \{1, \dots, 9\}$, where $c \in \mathbb{F}_{q^2}$, and each of μ and ϕ is one of x , y , or z . Then we raise to the power $q^3 - q^2$ and take the Teichmüller lifting to get

$$\chi_{\overline{H}} \equiv \sum_{k_1+\dots+k_9=q^3-q^2} \binom{q^3 - q^2}{k_1, \dots, k_9} \prod_{i=1}^9 T(v_i)^{k_i} \pmod{q^2}.$$

From Legendre's formula we have the p -adic valuation of the multinomial coefficient is

$$\frac{1}{p-1}(\sigma_p(k_1) + \dots + \sigma_p(k_9) - \sigma_p(q^3 - q^2)) = \frac{1}{p-1}(\sigma_p(k_1) + \dots + \sigma_p(k_9)) - t$$

where σ_p again indicates the p -adic digit sum.

We consider how the digits of k_i contribute to the λ -sums (1) of the monomial. Let $k_i = k_{i,0} + k_{i,1}p + \dots + k_{i,2t-1}p^{2t-1}$ and $v_i = c\mu\phi^q$. We can think of the digit $k_{i,j}$ as contributing once to λ_j (via μ) and once to λ_{j+t} (via ϕ , where the subscript is modulo $2t$). In fact, if there is no carry when we collect the exponents of x , y , and z , with $k_{i,j} = 0$, $i = 1, 2, 3$, whenever $j < 2t$, then

$$\begin{aligned} \lambda_j = \lambda_{j+t} &= \sum_{i=1}^9 (k_{i,j} + k_{i,j+t}), & 0 \leq j < t, \\ s_j &= s_{j+t}, & 0 \leq j < t, \\ \sigma(k_1) + \dots + \sigma(k_9) &= \lambda_0 + \dots + \lambda_{t-1} \\ &= \lambda_t + \dots + \lambda_{2t+1}. \end{aligned}$$

In this case, the lemma is a special case of (5) with equality throughout.

If there is a carry when we collect the exponents and reduce $(\pmod{q^2 - 1})$, say a carry from the $(j-1)^{\text{th}}$ place to the j^{th} place, then λ_{j-1} is reduced by p and λ_j is increased by 1, which means s_j is decreased by 1 and all the other s in the type are not affected. Since a carry can only increase the value of ξ and not decrease it, the lemma holds in this case too. \square

We have a complementary lemma for the Buekenhout-Metz case.

Lemma 8. Let $g = c_U T(x^{b'_1} y^{b'_2} z^{b'_3})$ be a term of $\chi_{\overline{U}}$ and let the \mathcal{H} -type of g_U be (s'_0, \dots, s'_{2t-1}) and let

$$\xi = |\{j \mid 0 \leq j \leq t-1, s_j = 2, \text{ and } s_{j+t} = 2\}|.$$

Then c_U is divisible by p^ξ .

Proof: We first consider the case that one of the exponents b'_1, b'_2 , or b'_3 is 0 or $(q^2 - 1)$. Then the other two exponents are either 0 or $(q^2 - 1)$, or else their sum is $(q^2 - 1)$. In these cases the λ -sums are all the same for $\lambda'_0, \dots, \lambda'_{2t-1}$. Using (6), the coefficient of $T(xyz)^{q^2-1}$ becomes the constant term, which must be divisible by $q = p^\beta$, by Proposition 3 and the discussion following (6). We are left with the cases that the \mathcal{H} -type of the monomial is either $(1, \dots, 1)$, or $(2, \dots, 2)$. If the \mathcal{H} -type is $(1, \dots, 1)$, there is nothing to prove. We already showed (see (7)) that the coefficients of monomials of \mathcal{H} -type $(2, \dots, 2)$ are divisible by q .

Now we consider terms in which b'_1, b'_2 , and b'_3 are all strictly between 0 and $q^2 - 1$. We find it convenient at this point to go back to the affine version of $\chi_{\overline{U}}$. That is, we assume $x = 1$ and do not write it. We have

$$(T(\alpha^q y^{2q}) - T(\alpha y^2) + T(\beta^q - \beta)T(y)^{q+1} - T(z)^q + T(z))^{2(q^2-q)}.$$

We remember that each term has a nonzero power of x . The only terms we have dropped are

$$(1 - T(x)^{q^2-1}) T(y)^{q^2-1},$$

which we have already considered. A typical term is

$$C \cdot D \cdot T(y)^{2qk_1+2k_2+qk_3+k_3} T(z)^{qk_4+k_5}, \quad (12)$$

where $C = (-1)^{k_2+k_4} T(\alpha)^{qk_1+k_2} T(\beta^q - \beta)^{k_3}$, $k_1 + \dots + k_5 = 2(q^2 - q)$, and $D = \binom{2(q^2 - q)}{k_1, \dots, k_5}$. We will show that the multinomial coefficient is divisible by p^ξ .

Recall that the p -adic valuation of D is the number of carries in $k_1 + \dots + k_5 = 2(q^2 - q)$. So if

$$k_i = h_{i,2t} p^{2t} + \dots + h_{i,1} p + h_{i,0}, \quad 1 \leq i \leq 5,$$

and c_j represents the carry from the $(j - 1)^{\text{th}}$ place to the j^{th} place, then

$$\begin{aligned} h_{1,0} + \cdots + h_{5,0} & -pc_1 = 0 \\ h_{1,j} + \cdots + h_{5,j} + c_j & -pc_{j+1} = 0 \quad \text{for } 0 < j < t \\ h_{1,t} + \cdots + h_{5,t} + c_t & -pc_{t+1} = p - 2 \\ h_{1,j} + \cdots + h_{5,j} + c_j & -pc_{j+1} = p - 1 \quad \text{for } t < j < 2t \\ h_{1,2t} + \cdots + h_{5,2t} + c_{2t} & = 1, \end{aligned}$$

and $\nu_p(D) = c_1 + \cdots + c_{2t}$. For some j , $1 \leq j \leq t$, assume that $c_j = c_{j+t} = 0$, so that this position does not contribute to $\nu_p(D)$. Clearly also $c_1 = \cdots = c_{j-1} = 0$ and we have $h_{i,0} = \cdots = h_{i,j-1} = 0$ for each i . Then

$$\sum_{\ell=0}^{j+t-1} p^\ell \sum_{i=1}^5 h_{i,\ell} = p^{t+j} - 2q;$$

$$(h_{1,j} + \cdots + h_{5,j}) + \cdots + (h_{1,j+t-1} + \cdots + h_{5,j+t-1})p^{t-1} = q - 2p^{t-j};$$

$$(h_{1,j+t} + \cdots + h_{5,j+t}) + \cdots + (h_{1,2t} + \cdots + h_{5,2t})p^{t-j} = 2p^{t-j} - 1.$$

Adding the last two expressions, multiplying by $(q + 1)$, grouping the terms, and using the formula for the twisted degrees of each k_i , $i = 1, \dots, 5$, it immediately follows that

$$\sum_{i=1}^5 \left((p^{t-j}k_i \mod q^2 - 1) + (p^{2t-j}k_i \mod q^2 - 1) \right) = q^2 - 1, \quad (13)$$

where each of the terms in (13) is reduced before adding.

We want to decide whether s_j and s_{j+t} can both be 2 in the \mathcal{H} -type of this monomial. From (2) we have, since $b'_1 < q^2 - 1$,

$$\begin{aligned} E &= (q^2 - 1)(s_j + s_{j+t} - 2) \\ &= \sum_{i=1}^3 (p^{2t-j}b'_i \mod (q^2 - 1) + p^{t-j}b'_i \mod (q^2 - 1)) \\ &\quad - 2(q^2 - 1) \\ &< \sum_{i=2}^3 p^{2t-j}b'_i \mod (q^2 - 1) + \sum_{i=2}^3 p^{t-j}b'_i \mod (q^2 - 1). \end{aligned}$$

Substituting for b'_2 and b'_3 from (12), distributing the $(\text{mod } q^2 - 1)$ operation, and using (13), we have

$$\begin{aligned}
 E &< p^{2t-j}(2qk_1 + 2k_2 + qk_3 + k_3) \pmod{q^2 - 1} \\
 &+ p^{t-j}(2qk_1 + 2k_2 + qk_3 + k_3) \pmod{q^2 - 1} \\
 &+ p^{2t-j}(qk_4 + k_5) \pmod{q^2 - 1} \\
 &+ p^{t-j}(qk_4 + k_5) \pmod{q^2 - 1} \\
 &\leq 2 \sum_{i=1}^3 \left((p^{t-j}k_i \pmod{q^2 - 1}) + (p^{2t-j}k_i \pmod{q^2 - 1}) \right) \\
 &+ \sum_{i=4}^5 \left((p^{t-j}k_i \pmod{q^2 - 1}) + (p^{2t-j}k_i \pmod{q^2 - 1}) \right) \\
 &< 2(q^2 - 1),
 \end{aligned}$$

where the last inequality is again strict because we assumed $b_3 > 0$. We now have

$$s_j + s_{j+t} < 4.$$

and the lemma is proved. \square

With Lemma 7 and Lemma 8 we have proved the conjecture of Baker and Ebert in the Buekenhout-Metz case.

Theorem 9. *The number of points in the intersection of a Hermitian unital and a Buekenhout-Metz unital in $\text{PG}(2, q^2)$ is congruent to 1 modulo q .*

Numerical evidence suggests that the Tits unital (in desarguesian planes of order an odd power of two) also satisfy the conjecture of Baker and Ebert. In that case, Lemma 8 is not satisfied for individual terms of the expansion.

4. TWO HERMITIAN VARIETIES

Here we generalize Kestenband's result for two Hermitian unitals.

Theorem 10. *Let H_1 and H_2 be two nondegenerate Hermitian varieties in $\text{PG}(n, q^2)$. Then*

$$q^{n-1} \mid |H_1 \cap H_2|.$$

Proof: Let the \mathcal{H} -types of f and g be (s_0, \dots, s_{2t-1}) and $(n+1-s_0, \dots, n+1-s_{2t-1})$ and use (5). \square

REFERENCES

- [1] R. D. Baker and G. L. Ebert, “On Buekenhout-Metz unitals of odd order,” *J. Combin. Theory, Ser. A* **60** (1992), 67–84.
- [2] R. D. Baker and G. L. Ebert, “Intersection of unitals in the desarguesian plane,” *Congressus Numerantium* **70** (1990), 87–94.
- [3] M. Bardoe, P. Sin, The permutation modules for $\mathrm{GL}(n+1, \mathbb{F}_q)$ acting on $\mathbb{P}^n(\mathbb{F}_q)$ and \mathbb{F}_q^{n+1} , *J. London Math. Soc.* **61** (2000), 58–80.
- [4] Susan Barwick and Gary Ebert, *Unitals in Projective Planes*, Springer, 2008.
- [5] A. Blokhuis, A. E. Brouwer, H. Wilbrink, “Hermitian unitals are code words,” *Discrete Math.* **97** (1991), 63–68.
- [6] D. B. Chandler, P. Sin, and Q. Xiang, “The invariant factors of the incidence matrices of points and subspaces in $\mathrm{PG}(n, q)$ and $\mathrm{AG}(n, q)$,” *Transactions of the American Mathematical Society* **358** (2006), 4935–4957.
- [7] G. L. Ebert, “On Buekenhout-Metz unitals of even order,” *Europ. J. Combinatorics* **13** (1992), 109–117.
- [8] B. C. Kestenband, “Unital intersections in finite projective planes,” *Geom. Dedicata* **11** (1981), 107–117.
- [9] E. S. Lander, *Topics in algebraic coding theory*, D. Phil. Thesis, Oxford University, 1980.
- [10] R. Metz, “On a class of unitals,” *Geom. Dedicata* **8** (1979), 125–126.
- [11] D. Wan, “A Chevalley-Warning approach to p -adic estimates of character sums,” *Proceedings of the American Mathematical Society* **123** (1995), 45–54.

E-mail address: davidbchandler@gmail.com